

# Lorentz Gauge Theory and Spinor Interaction

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**Abstract:** A gauge theory of the Lorentz group, based on the different behavior of spinors and vectors under local transformations, is formulated in a flat space-time and the role of the torsion field within the generalization to curved space-time is briefly discussed.

The spinor interaction with the new gauge field is then analyzed assuming the *time gauge* and stationary solutions, in the non-relativistic limit, are treated to generalize the Pauli equation.

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## 1 Diffeomorphisms and Lorentz Transformations

The gauge freedom of the gravitational interaction arises from the Principle of General Covariance and it is represented by the invariance under diffeomorphisms (*diff.*'s). On the other hand, the equivalence of any local reference system deals with another gauge freedom expressed by local Lorentz transformations (*Ltr.*'s). The main point is that the former gauge freedom reabsorbs the latter implying the nonexistence of independent connections of the local Lorentz group (LG). In fact, spin connections become only a particular combination of the tetrad fields and their derivatives and the local LG loses its status of independent gauge transformation

$$x^\mu \rightarrow x^\mu + \alpha^\mu{}_\nu(x) x^\nu = x^\mu + \tilde{\alpha}^\mu(x) . \quad (1)$$

This scheme is well grounded in the case of scalar or macroscopic matter (spin-less matter) [1]. In fact, from a mathematical point of view, using the tetrad formalism, *diff.*'s act as a pullback on the tetrad, while local *Ltr.*'s as a rotation of the local basis; in the case of isometric coordinate transformations, however, the latter can be restated in terms of the former. Under an isometric *diff.*, spin connections transform like tensors, and cannot be gauge potentials for such *diff.*-induced local rotations.

Conversely, if we deal with spin-1/2 matter fields, local *Ltr.*'s can no way be reabsorbed in the group of *diff.*'s. This way, fermions give back to the LG its status of independent gauge

group since they are described by a spinor representation of the LG, while the *diff.* group does not admit any.

## 2 Spinors and Lorentz Gauge Theory in Flat Space-time

Using the tetrad formalism ( $e_\mu^a$  being vierbein vectors), the Lagrangian density of spin-1/2 fields on a  $4D$  flat manifold reads

$$\mathcal{L} = \frac{i}{2} \bar{\psi} \gamma^a e_\mu^a \partial_\mu \psi - \frac{i}{2} e_\mu^a \partial_\mu \bar{\psi} \gamma^a \psi - m \bar{\psi} \psi, \quad (2)$$

$\gamma_a$  being the Dirac matrices. Spinor fields have to recognize the isometric components of the *diff.* as a local *Ltr.* Therefore, let us now introduce a local *Ltr.*  $S = S(\Lambda(x))$ :  $\psi(x) \rightarrow S\psi(x)$  and assume that the  $\gamma$  matrices transform locally like vectors ( $S \gamma^a S^{-1} = (\Lambda^{-1})^a_b \gamma^b$ ). Taking into account the infinitesimal parameters  $\epsilon_b^a(x) \ll 1$ , we define the usual relations for the LG

$$S = I - \frac{i}{4} \epsilon^{ab} \tau_{ab}, \quad \tau_{ab} = \frac{i}{2} [\gamma_a, \gamma_b], \quad [\tau_{cd}, \tau_{ef}] = i \mathcal{F}_{cdef}^{ab} \tau_{ab}, \quad (3)$$

$\tau_{ab}$  being group generators and  $\mathcal{F}_{cdef}^{ab}$  the structure constants. In this scheme, the gauge invariance is restored by the covariant derivative

$$D_\mu \psi = (\partial_\mu - \frac{i}{4} A_\mu) \psi = (\partial_\mu - \frac{i}{4} A_\mu^{ab} \tau_{ab}) \psi, \quad (4)$$

where the gauge transformation  $\gamma^a e_\mu^a D_\mu \psi \rightarrow S \gamma^a e_\mu^a D_\mu \psi$  is assured by the gauge field  $A_\mu = A_\mu^{ab} \tau_{ab}$  which transforms like a natural Yang-Mill field associated to the LG.

The considerations developed above can be generalized to curved space-time. In [2], a geometrical interpretation to the new Lorentz gauge field is provided, which can be identified with the tetradic projection of the *contortion field* [3]. The connections of the theory split up into two different terms: spin connections, which restore the Dirac algebra in the physical space-time and gauge connections, which guarantee the invariance under local *Ltr.*'s, respectively. Comparing the results in flat space, gauge connections can be interpreted as real gauge fields of the local LG since they are non-vanishing quantities even in flat space-time, as requested for any gauge fields.

## 3 Generalized Pauli Equation in Flat Space-time

We now analyze the interaction of the  $4$ -spinor  $\psi$  with the new LG gauge field  $A_\mu$ . The implementation of the local Lorentz symmetry ( $\partial_\mu \rightarrow D_\mu$ ) in flat spaces, leads to the Lagrangian density

$$\mathcal{L} = \frac{i}{2} \bar{\psi} \gamma^a e_\mu^a \partial_\mu \psi - \frac{i}{2} e_\mu^a \partial_\mu \bar{\psi} \gamma^a \psi - m \bar{\psi} \psi + \frac{1}{8} e_\mu^a \bar{\psi} \{\gamma^c, \tau_{ab}\} A_\mu^{ab} \psi, \quad (5)$$

where  $\{\gamma^c, \tau_{ab}\} = 2 \epsilon_{abd}^c \gamma_5 \gamma^d$ . To study the interaction term, let us now start from

$$\mathcal{L}_{int} = \frac{1}{4} \bar{\psi} \epsilon_{abd}^c \gamma_5 \gamma^d A_c^{ab} \psi, \quad (6)$$

where  $a = \{0, i\}$ ,  $i = \{1, 2, 3\}$ . This allows us to split the gauge field into  $A_0^{0i}$ ,  $A_0^{ij}$ ,  $A_\gamma^{0i}$ ,  $A_\gamma^{ij}$ . We now impose the *time-gauge* condition  $A_0^{ij} = 0$  and neglect the term  $A_0^{0i}$  since it is summed over  $\epsilon_{0id}^0 \equiv 0$ . By these considerations, the total Lagrangian density can be rewritten as

$$\mathcal{L} = \frac{i}{2} (\bar{\psi} \gamma^a \partial_a \psi - \partial_a \bar{\psi} \gamma^a \psi) - m \bar{\psi} \psi + \bar{\psi} C_0 \gamma_5 \gamma^0 \psi + \bar{\psi} C_i \gamma_5 \gamma^i \psi, \quad (7)$$

with the identifications

$$C_0 = \frac{1}{4} \epsilon_{ij0}^k A_k^i, \quad C_i = \frac{1}{4} \epsilon_{0ji}^k A_k^{0j}, \quad (8)$$

where the component  $C_0$  is related to rotations, while  $C_i$  with boosts. Varying now the total action built up from the Lagrangian density (7) with respect to  $\psi^\dagger$ , we get the following equation

$$(i\gamma^0\gamma^0\partial_0 + C_i\gamma^0\gamma_5\gamma^i + i\gamma^0\gamma^i\partial_i + C_0\gamma^0\gamma_5\gamma^0)\psi = m\gamma^0\psi, \quad (9)$$

which is the eq. of motion for the 4-spinor  $\psi$  interacting with the Lorentz-gauge field described here by the fields  $C_0$  and  $C_i$ .

Let us now look for stationary solutions of the Dirac equation expressed by

$$\psi(\mathbf{x}, t) \rightarrow \psi(\mathbf{x}) e^{-i\mathcal{E}t}. \quad (10)$$

The 4-component spinor  $\psi(\mathbf{x})$  can be expressed in terms of two 2-spinors  $\chi(\mathbf{x})$  and  $\phi(\mathbf{x})$  by writing

$$\psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix}, \quad \psi^\dagger = (\chi^\dagger, \phi^\dagger), \quad (11)$$

furthermore, let us assume the standard representation for the Dirac matrices

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Using such expressions, the 2-component spinors  $\chi$  and  $\phi$  are found to satisfy two coupled equations (here we write explicitly the 3-momentum  $p^i$ )

$$(\mathcal{E} - \sigma_i \cdot C^i) \chi - (\sigma_i \cdot p^i + C_0) \phi = m \chi, \quad (12a)$$

$$(\mathcal{E} - \sigma_i \cdot C^i) \phi - (\sigma_i \cdot p^i + C_0) \chi = -m \phi. \quad (12b)$$

These equations will be used below to study the non-relativistic limit of the Dirac-coupled equation [4].

### The Non-Relativistic Limit

In order to investigate the low-energy limit, we write the spinor-field energy in the form

$$\mathcal{E} = E + m. \quad (13)$$

Substituting this expression in the system (12), the coupled equations rewrite now

$$(E - \sigma_i \cdot C^i) \chi = (\sigma_i \cdot p^i + C_0) \phi, \quad (14a)$$

$$(E - \sigma_i \cdot C^i + m) \phi = (\sigma_i \cdot p^i + C_0) \chi - m \phi. \quad (14b)$$

In the non-relativistic limit, both  $|E|$  and  $|\sigma_i \cdot C^i|$  terms are small in comparison with the mass term  $m$ . Then, equation (14b) can then be solved approximately as

$$\phi = \frac{1}{2m} (\sigma_i \cdot p^i + C_0) \chi. \quad (15)$$

It is immediate to see that  $\phi$  is smaller than  $\chi$  by a factor of order  $v/m$  (*i.e.*,  $v/c$  where  $v$  is the magnitude of the velocity). In this scheme, the 2-component spinors  $\phi$  and  $\chi$  form the so-called *small* and *large components*, respectively [5].

Substituting the expression of the small component (15) in (14a) and using standard Pauli matrices relations, we get

$$E \chi = \frac{1}{2m} [p^2 + C_0^2 + 2C_0(\sigma_i \cdot p^i) + \sigma_i \cdot C^i] \chi. \quad (16)$$

This equation exhibits a strong analogy with the electro-magnetic case and the so-called Pauli equation and can be used in the analysis of the energy levels as in the Zeeman effect. Let us now neglect the term  $C_0^2$  and introduce a Coulomb potential  $V(r)$ , through the substitution  $E \rightarrow E - V(r)$ , obtaining the expressions

$$H_0 = \frac{p^2}{2m} - \frac{Ze^2}{(4\pi\epsilon_0)r}, \quad H' = \frac{1}{2m} [2C_0(\sigma_i \cdot p^i) + \sigma \cdot C^i], \quad (17)$$

which characterize the electron dynamics in a hydrogen-like atom in presence of a gauge field of the LG. It is worth noting the presence of a term related to the helicity of the 2-spinor: this coupling is controlled by the rotation-like component associated to  $C_0$ . A Zeeman-like coupling associated to the boost-like component  $C_i$  is also present. The analysis of these interactions can be performed in a Stern-Gerlach thought experiment.

## References

- [1] N. Carlevaro, O.M. Lecian and G. Montani, *Ann. Fond. L. deBroglie* **32**(2-3) (2007).
- [2] O.M. Lecian and G. Montani, in preparation (2007).
- [3] F.W. Hehl, P. Heyde, G.D. Kerlick and J.M. Nester, *Rev. Mod. Phys* **48**, 393 (1976).
- [4] I.L. Shapiro, *Phys. Rep.* **357**, 113 (2002).
- [5] F. Mandl and G. Shaw, *Quantum field th.* (J. Wiley & Sons, Chichester (UK), 1998).